

Topological classification of systems of bilinear and sesquilinear forms*

Carlos M. da Fonseca[†] Vyacheslav Futorny[‡]
 Tetiana Rybalkina[§] Vladimir V. Sergeichuk[¶]

Abstract

Let \mathcal{A} and \mathcal{B} be two systems consisting of the same vector spaces $\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_t}$ and bilinear or sesquilinear forms $A_i, B_i : \mathbb{C}^{n_{k(i)}} \times \mathbb{C}^{n_{l(i)}} \rightarrow \mathbb{C}$, for $i = 1, \dots, s$. We prove that \mathcal{A} is transformed to \mathcal{B} by homeomorphisms within $\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_t}$ if and only if \mathcal{A} is transformed to \mathcal{B} by linear bijections within $\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_t}$.

AMS classification: 15A21, 37C15

Keywords: Topological classification; Bilinear and sesquilinear forms

1 Introduction

Two bilinear or sesquilinear forms $A, B : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ are *topologically equivalent* if there exists a homeomorphism (i.e., a continuous bijection whose inverse is also a continuous bijection) $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $A(u, v) = B(\varphi u, \varphi v)$ for all $u, v \in \mathbb{C}^n$. We prove that *two forms are topologically equivalent if and only if they are equivalent*; we extend this statement to arbitrary systems of forms. Therefore, the canonical matrices of bilinear and sesquilinear forms given in [10] are also their canonical matrices with respect to topological equivalence.

*Published in: Linear Algebra Appl. 515 (2017) 1–5.

[†]Department of Mathematics, Kuwait University, Kuwait, carlos@sci.kuniv.edu.kw

[‡]Department of Mathematics, University of São Paulo, Brazil, futorny@ime.usp.br

[§]Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine, rybalkina_t@ukr.net

[¶]Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine, sergeich@imath.kiev.ua

Two linear operators $A, B : \mathbb{C}^n \rightarrow \mathbb{C}^n$ are *topologically equivalent* if there exists a homeomorphism $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $B(\varphi u) = \varphi(Au)$, for all $u \in \mathbb{C}^n$. The problem of classifying linear operators up to topological equivalence is still open. It is solved by Kuiper and Robbin [11, 13] for linear operators without eigenvalues that are roots of unity. It is studied for arbitrary operators in [1, 2, 3, 8, 9] and other papers. The fact that the problem of topological classification of forms is incomparably simpler is very unexpected for the authors; three of them only reduce it to the nonsingular case in [5].

Hans Schneider [15] studies the topological space H_r^n of $n \times n$ Hermitian matrices of rank r and proves that its connected components coincide with the *-congruence classes. The closure graphs for the congruence classes of 2×2 and 3×3 matrices and for the *-congruence classes of 2×2 matrices are constructed in [4, 6]. The problems of topological classification of matrix pencils and chains of linear mappings are studied in [7, 14].

2 Form representations of mixed graphs

Let G be a *mixed graph*; that is, a graph that may have undirected and directed edges (multiple edges and loops are allowed); let $1, \dots, t$ be its vertices. Its *form representation* \mathcal{A} of dimension $\underline{n} = (n_1, \dots, n_t)$ is given by assigning to each vertex i the vector space $\mathbb{C}^{n_i} := \mathbb{C} \oplus \dots \oplus \mathbb{C}$ (n_i times), to each undirected edge $\alpha : i \text{ --- } j$ ($i \leq j$) a bilinear form $A_\alpha : \mathbb{C}^{n_i} \times \mathbb{C}^{n_j} \rightarrow \mathbb{C}$, and to each directed edge $\beta : i \rightarrow j$ a sesquilinear form $A_\beta : \mathbb{C}^{n_i} \times \mathbb{C}^{n_j} \rightarrow \mathbb{C}$ that is linear in the first argument and semilinear (i.e., conjugate linear) in the second. Two form representations \mathcal{A} and \mathcal{B} of dimensions \underline{n} and \underline{m} are *topologically isomorphic* (respectively, *linearly isomorphic*) if there exists a family of homeomorphisms (respectively, linear bijections)

$$\varphi_1 : \mathbb{C}^{n_1} \rightarrow \mathbb{C}^{m_1}, \dots, \varphi_t : \mathbb{C}^{n_t} \rightarrow \mathbb{C}^{m_t} \quad (1)$$

that transforms \mathcal{A} to \mathcal{B} ; that is,

$$A_\alpha(u, v) = B_\alpha(\varphi_i u, \varphi_j v), \quad u \in \mathbb{C}^{n_i}, v \in \mathbb{C}^{n_j} \quad (2)$$

for each edge $\alpha : i \text{ --- } j$ or $i \rightarrow j$.

Example. A form representation

$$\mathcal{A} : \quad A_\alpha \bigcirc \mathbb{C}^{n_1} \begin{array}{c} \xleftarrow{A_\beta} \\ \xrightarrow{A_\gamma} \end{array} \mathbb{C}^{n_2} \bigcirc A_\delta \quad (3)$$

of dimension $\underline{n} = (n_1, n_2)$ of the mixed graph

$$G : \quad \alpha \bigcirc 1 \begin{matrix} \xleftarrow{\beta} \\ \xrightarrow{\gamma} \end{matrix} 2 \bigcirc \delta \quad (4)$$

is a system consisting of the vector spaces $\mathbb{C}^{n_1}, \mathbb{C}^{n_2}$, bilinear forms $A_\alpha : \mathbb{C}^{n_1} \times \mathbb{C}^{n_1} \rightarrow \mathbb{C}$, $A_\beta : \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \rightarrow \mathbb{C}$, and sesquilinear forms $A_\gamma : \mathbb{C}^{n_2} \times \mathbb{C}^{n_1} \rightarrow \mathbb{C}$, $A_\delta : \mathbb{C}^{n_2} \times \mathbb{C}^{n_2} \rightarrow \mathbb{C}$. Two form representations \mathcal{A} and \mathcal{B} are topologically isomorphic (respectively, linearly isomorphic) if there exist homeomorphisms (respectively, linear bijections) φ_1 and φ_2 that transform \mathcal{A} to \mathcal{B} :

$$\begin{array}{ccc} \mathcal{A} : & A_\alpha \bigcirc \mathbb{C}^{n_1} & \begin{matrix} \xleftarrow{A_\beta} \\ \xrightarrow{A_\gamma} \end{matrix} \mathbb{C}^{n_2} \bigcirc A_\delta \\ & \downarrow \varphi_1 & \downarrow \varphi_2 \\ \mathcal{B} : & B_\alpha \bigcirc \mathbb{C}^{m_1} & \begin{matrix} \xleftarrow{B_\beta} \\ \xrightarrow{B_\gamma} \end{matrix} \mathbb{C}^{m_2} \bigcirc B_\delta \end{array}$$

Theorem. *Two form representations of a mixed graph are topologically isomorphic if and only if they are linearly isomorphic.*

3 Proof of the theorem

Lemma. *If $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is a homeomorphism, then $n = m$ and there exists a basis u_1, \dots, u_n of \mathbb{C}^n such that $\varphi u_1, \dots, \varphi u_n$ is also a basis of \mathbb{C}^n .*

Proof. Let $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ be a homeomorphism. By [12, Section 11], $n = m$. Let $k \in \{1, \dots, n-1\}$. Reasoning by induction, we suppose that there exist linearly independent vectors $u_1, \dots, u_k \in \mathbb{C}^n$ such that $v_1 := \varphi u_1, \dots, v_k := \varphi u_k$ are also linearly independent. Take $u \in \mathbb{C}^n$ such that u_1, \dots, u_k, u are linearly independent, and write $v := \varphi u$. If v_1, \dots, v_k, v are linearly independent, we set $u_{k+1} := u$.

Let v_1, \dots, v_k, v be linearly dependent. Take a nonzero $w \in \mathbb{C}^n$ that is orthogonal to v_1, \dots, v_k . Then $v_1, \dots, v_k, v + aw$ are linearly independent for each nonzero $a \in \mathbb{C}$. Write $u(a) := \varphi^{-1}(v + aw)$ and consider the matrix $M(a)$ with columns $u_1, \dots, u_k, u(a)$. Since $\text{rank } M(0) = k+1$, there is a $(k+1) \times (k+1)$ submatrix $N(a)$ of $M(a)$ such that $\det N(0) \neq 0$. The determinant of $N(a)$ is a continuous function of a . Hence there is a nonzero $b \in \mathbb{C}$ such that $\det N(b) \neq 0$. Then $\text{rank } M(b) = k+1$. We take $u_{k+1} := u(b)$

and obtain linearly independent vectors u_1, \dots, u_{k+1} such that $\varphi u_1, \dots, \varphi u_{k+1}$ are also linearly independent.

We repeat this construction until we obtain the required u_1, \dots, u_n . \square

Proof of the theorem. Let \mathcal{A} and \mathcal{B} be two form representations of dimensions \underline{n} and \underline{m} of a mixed graph G .

If \mathcal{A} and \mathcal{B} are linearly isomorphic, then they are topologically isomorphic since $n_1 = m_1, \dots, n_t = m_t$ and each linear bijection $\varphi_i : \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_i}$ is a homeomorphism.

Let \mathcal{A} and \mathcal{B} be topologically isomorphic via a family of homeomorphisms (1). By the lemma, $n_1 = m_1, \dots, n_t = m_t$, and for each vertex $i \in \{1, \dots, t\}$ there exists a basis u_{i1}, \dots, u_{in_i} of \mathbb{C}^{n_i} such that $v_{i1} := \varphi u_{i1}, \dots, v_{in_i} := \varphi u_{in_i}$ is also a basis of \mathbb{C}^{n_i} . By (2), for each edge $\alpha : i \text{ --- } j$ or $i \longrightarrow j$ we have

$$A_\alpha(u_{ik}, u_{jl}) = B_\alpha(v_{ik}, v_{jl}), \quad k = 1, \dots, n_i, \quad l = 1, \dots, n_j.$$

Hence the matrix of $A_\alpha : \mathbb{C}^{n_i} \times \mathbb{C}^{n_j} \rightarrow \mathbb{C}$ in the bases u_{i1}, \dots, u_{in_i} and u_{j1}, \dots, u_{jn_j} is equal to the matrix of $B_\alpha : \mathbb{C}^{n_i} \times \mathbb{C}^{n_j} \rightarrow \mathbb{C}$ in the bases v_{i1}, \dots, v_{in_i} and v_{j1}, \dots, v_{jn_j} (each form is fully determined by its values on the basis vectors).

For each vertex $i \in \{1, \dots, t\}$, define the linear bijection $\psi_i : \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_i}$ such that $\psi_i u_{i1} = v_{i1}, \dots, \psi_i u_{in_i} = v_{in_i}$. The form representations \mathcal{A} and \mathcal{B} are linearly isomorphic via ψ_1, \dots, ψ_t . \square

Acknowledgement

C.M. da Fonseca was supported by Kuwait University Research Grant SM08/15. V. Futorny was supported by CNPq grant 301320/2013-6 and by FAPESP grant 2014/09310-5. V.V. Sergeichuk was supported by FAPESP grant 2015/05864-9.

References

- [1] S.E. Cappel, J.L. Shaneson, Linear algebra and topology, Bull. Amer. Math. Soc., New Series 1 (1979) 685–687.
- [2] S.E. Cappel, J.L. Shaneson, Nonlinear similarity of matrices, Bull. Amer. Math. Soc., New Series 1 (1979) 899–902.

- [3] S.E. Cappell, J.L. Shaneson, Non-linear similarity, *Ann. of Math.* 113 (2) (1981) 315–355.
- [4] A. Dmytryshyn, V. Futorny, B. Kågström, L. Klimenko, V.V. Sergeichuk, Change of the congruence canonical form of 2-by-2 and 3-by-3 matrices under perturbations and bundles of matrices under congruence, *Linear Algebra Appl.* 469 (2015) 305–334.
- [5] C.M. da Fonseca, T. Rybalkina, V.V. Sergeichuk, Topological classification of sesquilinear forms: Reduction to the nonsingular case, *Linear Algebra Appl.* 504 (2016) 581–589.
- [6] V. Futorny, L. Klimenko, V.V. Sergeichuk, Change of the $*$ congruence canonical form of 2-by-2 matrices under perturbations, *Electr. J. Linear Algebra* 27 (2014) 146–154.
- [7] V. Futorny, T. Rybalkina, V.V. Sergeichuk, Regularizing decompositions for matrix pencils and a topological classification of pairs of linear mappings, *Linear Algebra Appl.* 450 (2014) 121–137.
- [8] I. Hambleton, E.K. Pedersen, Topological equivalence of linear representations of cyclic groups. I, *Ann. of Math.* 161 (2005) 61–104.
- [9] I. Hambleton, E.K. Pedersen, Topological equivalence of linear representations for cyclic groups. II, *Forum Math.* 17 (2005) 959–1010.
- [10] R.A. Horn, V.V. Sergeichuk, Canonical forms for complex matrix congruence and $*$ congruence, *Linear Algebra Appl.* 416 (2006) 1010–1032.
- [11] N.H. Kuiper, J.W. Robbin, Topological classification of linear endomorphisms, *Invent. Math.* 19 (2) (1973) 83–106.
- [12] J. McCleary, *A First Course in Topology: Continuity and Dimension*, American Mathematical Society, Providence, RI, 2006.
- [13] J.W. Robbin, Topological conjugacy and structural stability for discrete dynamical systems, *Bull. Amer. Math. Soc.* 78 (1972) 923–952.
- [14] T. Rybalkina, V.V. Sergeichuk, Topological classification of chains of linear mappings, *Linear Algebra Appl.* 437 (2012) 860–869.
- [15] H. Schneider, Topological aspects of Sylvester’s theorem on the inertia of Hermitian matrices, *Amer. Math. Monthly* 73 (1966) 817–821.